Uniqueness of fixed point of a two-dimensional map obtained as a generalization of the renormalization group map associated to the self-avoiding paths on gaskets.

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ABSTRACT

Let $W(x,y) = a\,x^3 + b\,x^4 + f_5\,x^5 + f_6\,x^6 + (3\,a\,x^2)^2 y + g_5\,x^5 y + h_3\,x^3 y^2 + h_4\,x^4 y^2 + n_3\,x^3 y^3 + a_{24}\,x^2 y^4 + a_{05}\,y^5 + a_{15}\,x y^5 + a_{06}\,y^6$, and $X = \dfrac{\partial W}{\partial x}$, $Y = \dfrac{\partial W}{\partial y}$, where the coefficients are non-negative constants, with $a > 0$, such that $X^2(x,x^2) - Y(x,x^2)$ is a polynomial of x with non-negative coefficients.

Examples of the 2 dimensional map $\Phi : (x,y) \mapsto (X(x,y), Y(x,y))$ satisfying the conditions are the renormalization group (RG) map (modulo change of variables) for the restricted self-avoiding paths on the 3 and 4 dimensional pre-gaskets.

We prove that there exists a unique fixed point (x_f, y_f) of Φ in the invariant set $\{(x,y) \in {\mathbb{R}_+}^2 \mid x^2 \geqq y\} \setminus \{0\}$.

Keywords: renormalization group, fixed point uniqueness, self-avoiding paths, Sierpinski gasket

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1 Introduction and main results.

In this paper, we study existence and uniqueness of fixed point for a 2 dimensional discrete time dynamical system in the first quadrant ${\mathbb{R}_+}^2$, generated by the gradient

$$\Phi = (X, Y) = \mathrm{grad}\, W = (\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}) : \ {\mathbb{R}_+}^2 \to {\mathbb{R}_+}^2 \tag{1}$$

of a polynomial $W: \ {\mathbb{R}_+}^2 \to \mathbb{R}_+$ with non-negative coefficients, such that the set

$$\Xi = \{(x, y) \in {\mathbb{R}_+}^2 \mid y \leqq x^2\} \tag{2}$$

is an invariant set of $\Phi = (X, Y)$.

Let us state our main results. The first result deals with the existence of fixed point in the interior of Ξ.

Theorem 1 *Assume that $W: \ \mathbb{R}^2 \to \mathbb{R}$ satisfies the following:*

(i) W is a polynomial in 2 variables x and y, each term of which has positive coefficient and of total degree 3 or more. Moreover, the term x^3 exists (i.e., the coefficient of x^3 is non-zero).

(ii) Ξ of (2) is an invariant set of $\Phi = \mathrm{grad}\, W$: if $(x, y) \in \Xi$, then $\Phi(x, y) \in \Xi$. Moreover, $Y(x, x^2) < X(x, x^2)^2$, $x > 0$, for $\Phi = (X, Y)$.

(iii) There exsits a term of the form $x^n y$ in W, i.e., the coefficient of $x^n y$ is non-zero for some $n \geqq 2$.

(iv) $R(x, z) = X(x, x^2 z)^2 - Y(x, x^2 z)$ is a polynomial in z, $1 - z$, and x, with non-negative coefficients. Namely, there exists a polynomial $\tilde{R}(x, z, s)$ in 3 variables with non-negative coefficients such that $R(x, z) = \tilde{R}(x, z, 1 - z)$. Moreover,

$$\frac{R(x, z)}{Y(x, x^2 z)} = O(x), \quad x \to 0, \tag{3}$$

where $O(x)$ is uniform in $z \in [0, 1]$.

Then there exists a fixed point (x_f, y_f) of Φ in the interior $\Xi^o = \{(x, y) \in \mathbb{R}^2 \mid x > 0, \ y > 0, \ x^2 > y\}$ of Ξ. $\diamond$

We note that Theorem 1 is not a direct consequence of standard topological fixed point theorems on Ξ, which allows for a fixed point on the boundary of Ξ, $\partial \Xi = \{(x, 0) \mid x \geqq 0\} \cup \{(x, x^2) \mid x \geqq 0\}$, which is trivial, because $(0, 0)$ is a fixed point of Φ under the conditions in the Theorem. We are looking for a fixed point in Ξ^o, the interior of Ξ, not on the boundary.

We also note that restricting our attention to the subset $\Xi \subset {\mathbb{R}_+}^2$ is essential, because outside Ξ, fixed points may dissappear and appear with small changes in the coefficients of W. For example, let $W_\epsilon(x, y) = \frac{1}{3}x^3 + x^4 y + \epsilon y^6$. (This choice satisfies the conditions in Theorem 1 and Theorem 2 below for $0 \leqq \epsilon \leqq 8/3$.) Then for positive ϵ, there are 4 fixed points of $\Phi_\epsilon = \mathrm{grad}\, W_\epsilon$ in ${\mathbb{R}_+}^2$; $(0, 0)$, $(x_1, y_1) = (0.662 \cdots + O(\epsilon), 0.192 \cdots + O(\epsilon))$, $(0, (6\epsilon)^{-1/4})$, and one of order $(O(\epsilon^{1/8}), O(\epsilon^{-1/4}))$, while for $\epsilon = 0$ the last 2 are absent and we have only 2 fixed points.

An intuition for the specific conditions on W in Theorem 1 arises in an attempt to extend a corresponding simple fact for function with 1 variable. Let $f(x)$ be a polynomial with non-negative coefficients with lowest order term of x^3. Then there is a unique fixed point x_f of f' on the positive x axis ($f'(x_f) = x_f > 0$). Note that this is not the direct consequence of a standard topological fixed point theorem on an obvious invariant set $\mathbb{R}_+ = \{x \geqq 0\}$ of the map f', because $x = 0$ is a fixed point. Rather, the unique existence of the fixed point $x_f > 0$ is due to the positivity of the coefficients in the map and that a term x^n with $n > 1$ is smaller than x for small x and larger for large x.

A simple way of extending this fact to 2 variables would be to assume that the second variable y is of order x^2, at least for small x, and that this relation is preserved under the map $\Phi = (X, Y)$ in consideration. This motivates the non-negativity of the coefficients and conditions on $R = X^2 - Y$ in Theorem 1. We have added a couple of conditions to exclude fixed points on the boundary on $\partial\Xi \setminus \{0\}$ to avoid complications.

We turn to our second result, which is on the uniqueness of the fixed point (x_f, y_f) of Φ in Ξ^o. This is a more difficult problem than the existence result, and we have results only with 12 adjustable coefficients for W, in contrast to Theorem 1 which allows for indefinitely many terms.

Theorem 2 *Let $W : \mathbb{R}_+{}^2 \to \mathbb{R}_+$ be a polynomial defined by*

$$\begin{aligned} W(x,y) = {} & a\,x^3 + b\,x^4 + f_5\,x^5 + f_6\,x^6 + (3\,a\,x^2)^2 y + g_5\,x^5 y + h_3\,x^3 y^2 + h_4\,x^4 y^2 \\ & + n_3\,x^3 y^3 + a_{24}\,x^2 y^4 + a_{05}\,y^5 + a_{15}\,x y^5 + a_{06}\,y^6, \end{aligned} \tag{4}$$

where all the constants $a, b, f_5, f_6, g_5, h_3, h_4, n_3, a_{24}, a_{05}, a_{15}, a_{06}$, are non-negative, and $a > 0$, and $R(x,z) = X(x, x^2 z)^2 - Y(x, x^2 z)$ is a polynomial in z, $1-z$, and x, with non-negative coefficients, in the same sense as in the corresponding condition in Theorem 1. Then there exists a unique fixed point (x_f, y_f) of $\Phi = \operatorname{grad} W$ in Ξ^o. $\diamond$

The condition on R in Theorem 2 can be made explicit.

Proposition 3 *The conditions on W in Theorem 2 is equivalent to the following: W is as in (4), with the coefficients being non-negative, $a > 0$, and, $R_n \geqq 0$, $5 \leqq n \leqq 10$, where R_ns are*

$$\begin{aligned} R_5 &= 24\,a\,b - g_5 - 2\,h_3\,, \\ R_6 &= 16\,b^2 + 30\,a\,f_5 - 2\,h_4\,, \\ R_7 &= 216\,a^3 + 40\,b\,f_5 + 36\,a\,f_6 - 3\,n_3\,, \\ R_8 &= 288\,a^2\,b + 25\,f_5^2 + 48\,b\,f_6 + 30\,a\,g_5 + 18\,a\,h_3 - 5\,a_{05} - 4\,a_{24}\,, \\ R_9 &= 360\,a^2\,f_5 + 60\,f_5\,f_6 + 40\,b\,g_5 + 24\,b\,h_3 + 24\,a\,h_4 - 5\,a_{15}\,, \\ R_{10} &= 648\,a^4 + 216\,a^2\,f_6 + 18\,f_6^2 + 25\,f_5\,g_5 + 15\,f_5\,h_3 + 16\,b\,h_4 + 9\,a\,n_3 - 3\,a_{06}\,. \quad \diamond \end{aligned}$$

That this is necessary is easily seen, if one explicitly writes the coefficients of x^n in $R(x,1)$ for $5 \leqq n \leqq 10$. That the conditions in Proposition 3 are sufficient is proved by looking into the coefficients of x^n in $R(x,z)$ (each of which is a polynomial in z). It turns out that with W of the form (4), terms with x^n appear for $5 \leqq n \leqq 20$, among which no explicit negative signs appear for $n \geqq 11$, hence the condition hold automatically, and for the remaining $5 \leqq n \leqq 10$, the power of z in the terms with negative signs are larger than any of the terms with positive signs, hence with the non-negativity conditions at $z = 1$, assumed in Proposition 3, it is straightforward to find a polynomial in z and $1-z$ with non-negative coefficients. Proposition 3 is thus proved.

Among the examples of W satisfying the conditions in Theorem 2, or equivalently, in Proposition 3, are those related to the renormalization group (RG) map for the restricted self-avoiding paths on the 3 and 4 dimensional pre-gaskets [1, 3, 2]:

$$\begin{aligned}
W_3(x,y) &= \frac{1}{3}\,x^3+\frac{1}{2}\,x^4+\frac{2}{5}\,x^5+x^4y+2\,x^3y^2+\frac{22}{5}\,y^5,\\
W_4(x,y) &= \frac{\sqrt{3}}{9}\,x^3+\frac{1}{4}\,x^4+\frac{2\sqrt{3}}{15}\,x^5+\frac{1}{9}\,x^6+\frac{1}{3}\,x^4y\\
&+\frac{2\sqrt{3}}{9}\,x^5y+\frac{2\sqrt{3}}{9}\,x^3y^2+\frac{13}{18}\,x^4y^2+\frac{32\sqrt{3}}{81}\,x^3y^3+\frac{22}{27}\,x^2y^4\\
&+\frac{22}{135}\,y^5+\frac{44\sqrt{3}}{81}\,xy^5+\frac{31}{81}\,y^6.
\end{aligned} \tag{5}$$

It is straightforward to see that W_3 and W_4 satisfy all the conditions in Proposition 3. The fixed point equation $(x,y)=\Phi(x,y)$ for $\Phi=\operatorname{grad}W_3$ is

$$\begin{aligned}
x &= x^2+2x^3+2x^4+4x^3y+6x^2y^2,\\
y &= x^4+4x^3y+22y^4,
\end{aligned}$$

which coincides with that for $\vec{\Phi}$ in [1, (2.3) and (2.4)], and the fixed point equation $(x,y)=\Phi(x,y)$ for $\Phi=\operatorname{grad}W_4$ is, with the change of variables $x=\sqrt{3}\,x'$ and $y=3y'$,

$$\begin{aligned}
x' &= x'^2+3x'^3+6x'^4+6x'^5+12x'^3y'+30x'^4y'+18x'^2y'^2\\
&\quad+78x'^3y'^2+96x'^2y'^3+132x'y'^4+132y'^5,\\
y' &= x'^4+2x'^5+4x'^3y'+13x'^4y'+32x'^3y'^2+88x'^2y'^3+22y'^4+220x'y'^4+186y'^5,
\end{aligned}$$

which coincides with the fixed point equation for $\vec{\Phi}$ in [3, (33)] with $x\mapsto x'$ and $y\mapsto y'$. A motivation of the conditions in Theorem 2 was an attempt to generalize the known examples (5).

The class of W allowed by the conditions in Theorem 2 is a subset of that in Theorem 1. This may be easily seen from the following equivalent conditions to those in Theorem 2.

Proposition 4 *The conditions on W in Theorem 2 is equivalent to the following:*

(i) The conditions in Theorem 1 hold.

(ii) Each term has total degree no more than 6.

(iii) Terms containing positive powers of y has total degree 5 or 6.

(iv) xy^4 and x^2y^3 are absent. ◇

That these conditions imply those in Theorem 2 is easily seen, if one notices that the extra conditions in Proposition 4 implies (4) modulo the coefficient of x^4y, which is fixed by the condition $R(x,z)/Y(x,x^2z)=O(x)$. The converse is proved in a similar way.

In [1] and [3], the results in Theorem 1 and Theorem 2 are proved for $W=W_3$ and $W=W_4$ in (5), respectively, but the proofs there explicitly uses the explicit values of coefficients in W_3 and W_4. These values of the coefficients are essentially the numbers of certain figures (self-avoiding paths) on the 3 and 4 dimensional gaskets, respectively, and the non-Markovian nature of self-avoiding paths makes it hard to count these numbers, not to mention to find general formula for all d dimensional gaskets. It is therefore important

for the RG approach that the results in the above theorems could be derived from 'basic properties' for W which can be derived by simple arguments. It is not very difficult to derive the conditions in Proposition 4 (including those in Theorem 1) from basic graphical considerations in the case of restricted self-avoiding paths on 3 and 4 dimensional gaskets, so Theorem 2 provides a rather satisfactory alternate proofs to the corresponding original proofs in [1, 3], in that one no more needs to count the number of self-avoiding paths exactly, for a proof of existence and uniqueness of fixed points in Ξ^o.

We also note that the examples (5) do not seem to fit to any existing general theorems on fixed point uniqueness, much less the class in Theorem 2. This may reflect the fact that self-avoiding paths are non-Markovian and mathematically hard to analyze. The present study may then provide a new direction in the study of fixed point theorems.

A plan of this paper is as follows. In Section 2 we prove Theorem 1 and in Section 3 we prove Theorem 2.

We note that our proof for Theorem 2 in Section 3 in fact proves a stronger property than is stated in the Theorem, and does not hold for all W in the class satisfying the conditions in Theorem 1. However, it seems that even for examples where the proof in Section 3 breaks down, the statement in Theorem 2 still seems to hold. We therefore close this Introduction with the following conjecture.

Conjecture 5 *Uniqueness of the fixed point* (x_f, y_f) *of* $\Phi = \operatorname{grad} W$ *in* Ξ^o *hold under the condition in Theorem 1.* $\diamond$

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2 Proof of existence of fixed point.

Here we prove Theorem 1.

Let $W = W(x, y)$ be a polynomial with non-negative coefficients satisfying the conditons in Theorem 1, and define functions in 2 variables F and G by

$$G(x, z) = \frac{1}{x} X(x, x^2 z), \tag{6}$$

and

$$F(x, z) = z \frac{X^2(x, x^2 z)}{Y(x, x^2 z)}\,, \tag{7}$$

where $\operatorname{grad} W = (X, Y)$.

Note that with the change of variables $(x, y) \mapsto (x, z)$ defined by $y = x^2 z$, the set $\Xi \setminus \{0\} = \{(x, y) \in {\mathbb{R}_+}^2 \mid y \leqq x^2\} \setminus \{0\}$ is mapped to a strip in the first quadrant ${\mathbb{R}_+}^2$

$$\tilde{\Xi} = \{(x, z) \in \mathbb{R}^2 \mid x > 0,\ 0 \leqq z \leqq 1\}. \tag{8}$$

Since $W(x, y)$ has a term of a form $x^n y$ by a condition in Theorem 1, $Y(x, x^2 z) > 0$, $(x, z) \in \tilde{\Xi}$, hence $F(x, z)$ is well-defined, positive and analytic on $\tilde{\Xi}$.

Note also that $(x, x^2 z) \in \Xi \setminus \{0\}$ is a fixed point of $\Phi = (X, Y)$ if and only if $(x, z) \in \tilde{\Xi}$ and $F(x, z) = G(x, z) = 1$.

Lemma 6 *$G(x,z)$ is a polynomial in x and z with non-negative coefficients, satisfying*

$$G(x,z) = 3ax\,(1+O(x)), \tag{9}$$

where a is the coefficient of the term x^3 in W. Furthermore, the contour set for $G=1$ in the strip $\tilde{\Xi}$ is a smooth curve connecting the floor $z=0$ and the ceiling $z=1$; Namely, there exists a positive continously differentiable function $x^*(z)>0$ for $0 \leqq z \leqq 1$ such that

$$\{(x,z)\in\tilde{\Xi} \mid G(x,z)=1\} = \{(x^*(z),z) \mid 0 \leqq z \leqq 1\}. \tag{10}$$

F satisfies $F(x,z)>0$, $(x,z)\in\tilde{\Xi}^o$, $F(x,0)=0$ and $F(x,1)>1$ for $x>0$. ◇

Proof. All the statement about G is obvious from the conditions in Theorem 1, except perhaps the last one. To see that the stated x^* exists, first note that by the conditions in Theorem 1, W is a polynomial with non-negative coefficients with lowest order being x^3, hence G is a polynomial with non-negative coefficients satisfying $\dfrac{\partial G}{\partial x}(x,z) \geqq 3a > 0$, $\lim_{x\downarrow 0} G(x,z)=0$, and $\lim_{x\to\infty} G(x,z)=\infty$ for $0\leqq z\leqq 1$. This with an implicit function theorem implies that there uniquely exists a continuously differentiable function $x^*:\ [0,1]\to\mathbb{R}_{>0}$ such that $G(x^*(z),z)=1$, $0\leqq z\leqq 1$, and monotonicity of G in x implies that all the point satisfying $G(x,z)=1$ is on the curve $\{(x^*(z),z)\}$.

Statements on F are also easy, if one notes

$$F(x,z) = z\left(1+\frac{R(x,z)}{Y(x,x^2z)}\right). \tag{11}$$

□

A proof of Theorem 1 is now obvious, because Lemma 6 implies $F(x^*(0),0)=0$ and $F(x^*(1),1)>1$, for a smooth curve $\{(x^*(z),z)\mid 0\leqq z\leqq 1\}\subset\tilde{\Xi}$ hence there is a $z^*\in(0,1)$ such that $F(x^*(z^*),z^*)=G(x^*(z^*),z^*)=1$, which, as noted at the beginning of this section, implies the existence of a fixed point of $\operatorname{grad} W=(X,Y)$ in Ξ^o. □

3 Proof of uniqueness of fixed point.

Here we prove Theorem 2.

Let $J_{GF}(x,z)$ be the Jacobian matrix of the map $(x,z)\mapsto(G(x,z),F(x,z))$;

$$J_{GF} = \frac{\partial G}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial z}. \tag{12}$$

A core of our proof of Theorem 2 is to prove $J_{GF}\neq 0$ on the contour curve $G=1$ in $\tilde{\Xi}$. This implies that the map is locally one-to-one, which further implies, with additional properties such as (10) and Lemma 7 below, global one-to-one properties, implying uniqueness of the fixed point.

The proof of Theorem 2 in this section starts with Lemma 7, then follows Lemma 8 where we prove that, with these properties, positivity of Jacobian J_{GF} is sufficient for a proof of Theorem 2. Up to this pont, the arguments are 'soft' and all the results hold for the class in Theorem 1. The hardest part comes last, a proof that $J_{GF}>0$ for the class

assumed in Theorem 2. That this is hard may be seen if one notices that outside $\tilde{\Xi}$ there may be more than one fixed points (as are the cases for the examples below Theorem 1 and (5) [1, 3]), hence $J_{GF} < 0$ actually occurs for some $(x, z) \in {\mathbb{R}_+}^2$. We must therefore find a nice quantity which is explicitly positive only in a subset of $\tilde{\Xi}$ and then prove (as we will in Lemma 9) that the quantity is a lower bound of J_{GF} using inequalities in Proposition 3.

Lemma 7 *Under the conditions in Theorem 1, if $x > 0$ is sufficiently small, then*

$$J_{GF}(x, z) > 0, \quad 0 \leqq z \leqq 1,$$

and furthermore, the set of $(x, z) \in \tilde{\Xi}$ satisfying $F(x, z) = 1$ is a single curve for small x, having $(0, 1)$ as an endpoint. (More precisely, there exists $\delta > 0$ such that $\{(x, z) \in (0, \delta] \times [0, 1] \mid F(x, z) = 1\}$ is a curve whose endpoints are $(0, 1)$ and a point on $x = \delta$.) $\diamond$

Proof. (11) and (3) imply

$$F(x, z) = z\,(1 + O(x)), \tag{13}$$

uniformly in $0 \leqq z \leqq 1$, which, with (9) implies $J_{GF} = a + O(x) > 0$ for small x, say $0 < x < \delta$. This in particular implies $\operatorname{grad} F \neq 0$, hence $\{(x, z) \in (0, \delta] \times [0, 1] \mid F(x, z) = 1\}$ is a finite union of non-intersecting smooth curves, each segment of which is either closed or with endpoints at $x = 0$ or $x = \delta$.

Lemma 6 implies $F(x, 1) > 1$ and $F(x, 0) = 0$, so that $F = 1$ cannot intersect $z = 0$ nor $z = 1$. Also (13) implies $F(+0, z) = z$, which, with $F(x, 1) > 1$ implies that the contour curve for $F = 1$ exists and intersect $x = 0$ at $z = 1$. By definition,

$$F(x, 1-u) - 1 = (1-u)\,\frac{X^2}{Y}(x, x^2\,(1-u)) - 1 = \frac{R(x, 1-u) - uX^2(x, x^2\,(1-u))}{Y(x, x^2\,(1-u))}\,,$$

and

$$\begin{aligned} &R(x, 1-u) - uX^2(x, x^2\,(1-u)) = R(x, 1) - 9a^2x^4\,(1 + O(x))u + x^7\,O_x(u^2) \\ &= O(x^5) - 9a^2x^4\,(1 + O(x))u. \end{aligned}$$

Hence, For small x and u, the contour $F(x, 1-u) = 1$ is uniquely given by $u = O(x)$ in $(x, 1-u) \in (0, \delta] \times [0, 1]$. $\square$

Lemma 8 *In addition to the conditions in Theorem 1, assume that $J_{GF}(x, z) \neq 0$ on*

$$\tilde{\Xi}' = \{(x, z) \in (0, \infty) \times (0, 1) \mid G(x, z) \leqq 1,\ F(x, z) \leqq 1\} \quad \subset \tilde{\Xi}^o. \tag{14}$$

Then the fixed point of $\Phi = \operatorname{grad} W = (X, Y)$ is unique in Ξ^o. $\diamond$

Proof. As noted at the beginning of Section 2, $(x, y) \in \Xi^o$ is a fixed point of Φ if and only if $(x, y/x^2) \in \tilde{\Xi}^o$ and $F(x, y/x^2) = G(x, y/x^2) = 1$. Theorem 1 implies that there is a fixed point $(x_f, y_f) \in \Xi^o$ of Φ. Put $z_f = y_f/x_f^2$.

Then $F(x_f, z_f) = G(x_f, z_f) = 1$, hence $(x_f, z_f) \in \tilde{\Xi}'$.

As in the proof of Lemma 7, $J_{GF} \neq 0$ implies $\operatorname{grad} F \neq 0$, which further implies that

$$A = \{(x, z) \in \tilde{\Xi}' \mid F(x, z) = 1\} \quad \subset \tilde{\Xi}'$$

is a finite union of non-intersecting smooth curves, each segment of which is either closed, or with one endpoint $(0,1)$ and the other on $\{(x,z) \in \tilde{\Xi}' \mid G(x,z) = 1\}$. (x_f, z_f) is on one of such curves.

Suppose that (x_f, z_f) is on a smooth curve $C:\ [0,1] \to A$ with $\frac{dC}{ds}(s) \neq 0$, $0 < s < 1$, whose endpoints are both on $G = 1$; $G(C(0)) = G(C(1)) = 1$ and $C(s) \in \tilde{\Xi}'$, $0 \leqq s \leqq 1$. A mean-value Theorem then implies that there exists $s_0 \in (0,1)$ such that

$$0 = \left.\frac{dG(C(s))}{dds}\right| = (\mathrm{grad}\, G)(C(s_0)) \cdot \frac{dC}{ds}(s_0).$$

Multiplying by $(\frac{\partial F}{\partial z}, -\frac{\partial F}{\partial x})(C(s_0))$ from left, we have $0 = J_{GF}(C(s_0))\frac{dC}{ds}(s_0)$, which contradicts the assumption $J_{GF} \neq 0$ on $\tilde{\Xi}'$. Therefore, a contour curve in A on which (x_f, z_f) exists, cannot have both endpoints on $G = 1$. Similarly, such a curve cannot be a closed curve in $\tilde{\Xi}'$. Therefore the curve must have one endpoint $(0,1)$ and the other on $G = 1$, the latter endpoint being the point (x_f, z_f).

By Lemma 7, a curve of the contour set $F = 1$ that has endpoint $(0,1)$ is unique. Therefore there is only one curve $C \subset A \subset \tilde{\Xi}'$ on which there is a point satisfying $F(x_f, z_f) = G(x_f, z_f) = 1$, hence, as noted at the beginning of the proof, the fixed point $(x_f, x_f^2 z_f)$ is unique in Ξ^o. $\square$

A proof of Theorem 2 is now reduced to proving $J_{GF}(x,z) \neq 0$ on (14) under the conditions in Theorem 2. This follows as the direct consequence of the following Lemma 9. In fact, the Lemma states positivity of $J_{GF}(x,z)$ on

$$\tilde{\Xi}'' = \{(x,z) \in (0,\infty) \times (0,1) \mid F(x,z) \leqq 1\} \quad \subset \tilde{\Xi}^o, \tag{15}$$

which is larger than (14).

Lemma 9 *Assume that W satisfies the conditions in Theorem 2. Let e be a function defined by*

$$e(x,z) = (1-z)x^2 \frac{Y^2}{X^2}(x, x^2 z) \left(J_{GF} - \frac{F(1-F)}{z(1-z)} \frac{\partial G}{\partial x} \right)(x,z). \tag{16}$$

Then $e(x,z)$ is a polynomial in x, z, $1-z$ with non-negative coefficients. Namely, there exists a polynomial $f(x,z,s)$ in 3 variables with non-negative coefficients such that $e(x,z) = f(x,z,1-z)$. Furthermore, $f(x,z,s)$ has a term $a^4\, s^2 z x^9$, hence in particular, $e(x,z) > 0$, and consequently, $J_{GF}(x,z) > 0$, $(x,z) \in \tilde{\Xi}''$. $\diamond$

Proof. The last claim is by explicit calculation of order x^9 terms. For this and for the calculations below, we use Mathematica software to assist the simple algebraic manupulation such as expanding and factoring. ($e(x,z)$ has more than 300 terms with positive coefficients and more than 80 terms with negative ones!)

The problem is to use R_ns in Proposition 3 to eliminate apparent negative signs in $e(x,z)$. It turns out that we have an expression

$$e(x,z) = e_c(x,z,1-z) + e_r(x,z,1-z) \tag{17}$$

where

$e_c(x,z,s) = 3\,a\,R_5\,z\,x^7 + 3\,a\,R_6\,z\,x^8 + 8\,b\,R_5\,z\,x^8 + 3\,a\,R_7\,z^2\,(1+s)\,x^9 + 8\,b\,R_6\,z\,x^9 +$
$15\,f_5\,R_5\,z\,x^9 + 3\,a\,R_8\,z^3\,(1+2\,s)\,x^{10} + 8\,b\,R_7\,z^2\,(1+s)\,x^{10} + 15\,f_5\,R_6\,z\,x^{10} +$
$R_5\,(a^2\,(144\,z^3+36\,z^3\,s)+f_6\,24\,z)\,x^{10} + 8\,b\,R_8\,z^3\,(1+2\,s)\,x^{11} + 15\,f_5\,R_7\,z^2\,(1+s)\,x^{11} +$
$R_6\,(6\,a^2\,z^3\,(18+6\,z+6\,z\,s)+f_6\,24\,z)\,x^{11} + R_5\,(g_5\,(3\,z^2+45\,z^2\,s+6\,z^4+16\,z^5)+h_3\,(4\,z^2+$
$12\,z^2\,s^2+11\,z^5)+a\,b\,24\;s^2\,z^2\,(8+z))\,x^{11} + R_9\,(3\,z^4+9\,z^4\,s)\,a\,x^{11} + R_{10}\,a\,6\;(1+4\,s)\,z^5\,x^{12} +$
$R_9\,b\,8\,(1+3\,s)\,z^4\,x^{12} + R_7\,f_6\,24\,(1+s)\,z^2\,x^{12} + R_8\,f_5\,(1+2\,s)\,z^3\,15\,x^{12} + R_7\,a^2\,18\,z^3\,(3+$
$4\,z+z^2+4\,z^2\,s)\,x^{12} + R_6\,h_3\,(6\,z^4+9\,z^5)\,x^{12} + R_6\,g_5\,5\,z^2\,(9\,s+z^2+5\,z^2\,s+4\,z^4)\,x^{12} +$
$R_5\,h_4\,4\,z^3\,(1+8\,s+z^2+4\,z^3)\,x^{12} + R_9\,f_5\,15\,(1+3\,s)\,z^4\,x^{13} + R_{10}\,b\,16\,(1+4\,s)\,z^5\,x^{13} +$
$R_8\,f_6\,24\,z^3\,(1+2\,s)\,x^{13} + R_8\,a^2\,9\,z^4\,(16+4\,(1+z)\,s)\,x^{13} + R_7\,g_5\,(25\,z^3\,s+25\,z^3\,(1+z)\,s+$
$25\,z^6)\,x^{13} + R_5\,n_3\,3\,z^3\,(10\,z\,s+7\,z^3)\,x^{13} + R_7\,h_3\,(5\,z^5+10\,z^6)\,x^{13} + R_5\,a\,f_6\,108\,z^3\,s^3\,x^{13} +$
$R_6\,h_4\,(16\,z^4+8\,z^4\,(1+z)\,s+8\,z^7)\,x^{13} + R_{10}\,f_5\,30\,z^5\,(1+4\,s)\,x^{14} + R_9\,f_6\,24\,(1+3\,s)\,z^4\,x^{14} +$
$R_9\,a^2\,18/5\,(39+46\,s)\,z^5\,x^{14} + R_6\,n_3\,(21\,z^5+3\,z^6)\,x^{14} + R_7\,h_4\,22\,z^5\,x^{14} + R_8\,g_5\,(25\,z^4+$
$8\,z^4\,s+6\,z^4\,(1+z)\,s)\,x^{14} + R_8\,h_3\,3\,(5+3\;s)\,z^6\,x^{14} + R_5\,a_{24}\,(10\,z^5+6\,z^6)\,x^{14} +$
$R_{10}\,f_6\,48\,z^5\,(1+4\,s)\,x^{15} + R_{10}\,a^2\,288\,(1+2\,s)\,z^6\,x^{15} + R_9\,h_3\,18\,z^5\,x^{15} + R_5\,a_{15}\,12\,z^6\,x^{15} +$
$R_9\,g_5\,2/5\,(61+94\,s)\,z^5\,x^{15} + R_6\,a_{24}\,24\,z^5\,x^{15} + R_8\,h_4\,z^5\,(4+20\,z+20\,z\,s)\,x^{15} +$
$R_7\,n_3\,21\,z^6\,x^{15} + R_8\,n_3\,z^7\,(21+9\,s)\,x^{16} + R_{10}\,g_5\,(50\,z^6+120\,z^6\,s)\,x^{16} + R_{10}\,h_3\,(30\,z^7+$
$24\,z^7\,s)\,x^{16} + R_9\,h_4\,(23\,z^6+8\,z^6\,s)\,x^{16} + R_7\,a_{24}\,26\,z^6\,x^{16} + R_6\,a_{15}\,13\,z^6\,x^{16} +$
$R_{10}\,h_4\,(48\,z^7+64\,z^7\,s)\,x^{17} + R_9\,n_3\,(17\,z^7+z^8+7\,z^8\,s)\,x^{17} + R_8\,a_{24}\,16\,z^8\,x^{17} + R_7\,a_{15}\,(2\,z^8+$
$12\,z^9)\,x^{17} + R_{10}\,n_3\,(42\,z^8+24\,z^8\,s)\,x^{18} + R_9\,a_{24}\,(3\,z^8+14\,z^9)\,x^{18} + R_8\,a_{15}\,(10\,z^8+$
$5\,z^9)\,x^{18} + R_{10}\,a_{24}\,32\,z^9\,x^{19} + R_9\,a_{15}\,(z^9+8\,z^{10})\,x^{19} + R_{10}\,a_{15}\,(10\,z^{10}+8\,z^{11})\,x^{20}\,,$

with which the remainder $e(x,z) - e_c(x,z,1-z)$ has an expression $e_r(x,z,1-z)$ for a polynomial $e_r(x,z,s)$ with non-negative coefficients, where non-negativity of e_r holds by non-negativity of $a, b, \cdots$, without the conditions $R_n \geqq 0$. For completeness, we give an explicit form of e_r in the appendix.

By (17), $e(x,z) = f(x,z,1-z)$ with $f(x,z,s) = e_c(x,z,s) + e_r(x,z,s)$, which completes a proof of Lemma 9, hence, as noted at the beginning of this section, a proof of Theorem 2 is also complete. $\square$

We remark that Lemma 9 is a result much stronger than is required to prove Theorem 2. In fact, with Lemma 9, a similar argument as for the contour curves $F=1$ and $G=1$ hold for contours $G=c$ for any $c>0$ and $F=c$ for any $c \leqq 1$, hence in particular, we have the following.

Corollary 10 *$\tilde{\Xi}''$ defined by (15) is a connected set, whose boundary is $\{x=0\} \cup \{z=0\} \cup \{F=1\}$, and the map $(x,z) \mapsto (G,F)$ is globally one-to-one on $\tilde{\Xi}''$.* $\diamond$

We also remark that the formula (16) and the rather lengthy e_c was found to work by trial and error, and it is an open problem to find their intuitive (either mathematical or physical) meaning.

A An explicit form of e_r.

For completeness, we will give an explicit form of e_r defined in the proof of Lemma 9. (Note that it is not unique. For example, there are more than 1 ways of writing $3-2z$ as a polynomial of z and $1-z$ with positive coefficients; $3-2z = 1+2(1-z) = z+3(1-z)$.)

$e_r(x,z,s) = \sum_{n=9}^{30} C[n,z,s]\,x^n$, where,

$C[9,z,s] = 12\,a\,(54\,a^3 + 10\,b\,f_5 + 9\,a\,f_6)\,s^2\,z,$

$C[10,z,s] = s^2\,z\,(320\,b^2\,f_5 + 144\,a\,b\,(18\,a^2\,z + f_6\,(3+2\,z)) + 3\,a\,(25\,f_5^2\,(1+2\,z) + 3\,(22\,a\,g_5 + 16\,a\,g_5\,z + 4\,a\,h_3\,z + 5\,a_{05}\,z^2))),$

$C[11,z,s] =$
$s^2\,z\,(15\,g_5^2 + 38\,g_5^2\,z + 110\,g_5\,h_3\,z + 32\,h_3^2\,z + 16\,g_5^2\,z^2 + 43\,g_5\,h_3\,z^2 + 22\,h_3^2\,z^2 + 144\,a^2\,h_4\,z\,(1+z) + 2160\,a^3\,f_5\,z\,(1+2\,z) + 384\,b^2\,(f_6 + 2\,f_6\,z) + 180\,a\,f_5\,f_6\,(4+2\,z+3\,z^2) + 40\,b\,(3\,a_{05}\,z^2 + 10\,f_5^2\,(2+z))),$

$C[12,z,s] = s^2\,z\,(225\,a_{05}\,f_5\,z^2 + 432\,b^2\,h_3\,z^2 + 360\,a\,f_5\,h_3\,z^3 + f_5^3\,(375 + 750\,z) + b\,f_5\,f_6\,(2160 + 2400\,z + 1440\,z^2) + b^2\,g_5\,(400\,z^2 + 320\,z^2\,s) + h_3\,h_4\,(90\,z^2 + 32\,z^3) + g_5\,h_4\,(104\,z + 66\,z^2 + 56\,z^3) + a\,f_6^2\,(972 + 216\,z + 324\,z^2 + 432\,z^3) + a^2\,b\,f_5\,(12960\,z + 10800\,z^2 + 2880\,z^3) + a^3\,f_6\,(1296\,z + 5832\,z^2 + 7776\,z^3) + a^5\,(23328\,z^2 + 31104\,z^3)),$

$C[13,z,s] = z\,(72\,a\,b\,n_3\,z^5 + a_{05}\,f_6\,360\,s^2\,z^2 + f_5^2\,f_6\,300\,s^2\,(5+10\,z+9\,z^2) + a^3\,g_5\,1080\,z^4\,(1+4\,s) + a^3\,h_3\,216\,z^3\,(3+s\,(12+7\,z)) + a\,f_6\,g_5\,36\,z\,(5\,z^4 + 25\,z^3\,s + 35\,s^2 + s^3\,(15+8\,z)) + a\,f_6\,h_3\,36\,z^3\,((3+16\,z)\,s + 3) + a^2\,a_{24}\,(36\,z^4 + 144\,z^4\,s) + h_3\,n_3\,72\,s^2\,z^3 + g_5\,n_3\,3\,s\,z^2\,(32\,s\,(1+z) + 2 + 3\,z) + b\,f_5\,h_3\,80\,s\,z^2\,(5+s\,(10+7\,z)) + b\,f_5\,g_5\,200\,s^2\,z\,(10+10\,z+3\,z^2) + a^2\,a_{05}\,900\,s^2\,z^3 + a^2\,f_5^2\,900\,s^2\,z\,(10+20\,z+z^2) + b\,f_6^2\,288\,s^2\,(5+10\,z+3\,z^2+4\,z^3) + a^2\,b\,f_6\,864\,s^2\,z\,(20+13\,z+21\,z^2) + a^4\,b\,51840\,s^2\,z^2\,(1+z) + a\,f_5\,h_4\,120\,z^2\,(1 + 2\,z^3\,s + 9\,s^2) + b^2\,h_4\,128\,s^2\,z^2\,(5+6\,z+z\,s) + h_4^2\,16\,s^2\,z^2\,(1+z+z^2)),$

$C[14,z,s] = z/5\,(a\,f_6\,h_4\,360\,z^2\,(3+6\,s+13\,s^2) + a\,a_{24}\,b\,(720\,z^4 + 720\,z^4\,s) + b^2\,n_3\,240\,z^3\,(3+s\,(8+z)) + a_{24}\,g_5\,90\,z^4\,s + h_4\,n_3\,30\,z^3\,s^2 + a\,g_5^2\,75\,z^2\,(5\,(1+s)^2 + s^2\,(35+12\,z)) + a_{05}\,g_5\,975\,s^2\,z^3 + b\,f_6\,g_5\,240\,z\,(4\,z^3 + 6\,z^4 + 46\,s + (9+64\,z)\,s^2) + a\,f_5\,n_3\,4950\,s^2\,z^3 + b\,f_5\,h_4\,400\,z^2\,(3\,z^2 + 2\,z^3 + 22\,s + 22\,z\,s^2) + f_5\,f_6^2\,900\,s^2\,(11+22\,z+33\,z^2+12\,z^3) + f_5^2\,g_5\,125\,s^2\,z\,(55+110\,z+126\,z^2) + a^3\,h_4\,864\,z^3\,(8+24\,s+23\,s^2) + a^2\,b\,g_5\,720\,z^2\,(21\,z^3 + 105\,s + (5+37\,z)\,s^2) + a^2\,f_5\,f_6\,1080\,z\,(z^4 + z^2\,5\,s + 55\,s^2\,(2+4\,z+3\,z^2)) + a^4\,f_5\,6480\,z^2\,(z^3 + z^2\,5\,s + 55\,(1+2\,z)\,s^2) + a_{24}\,h_3\,180\,z^5\,s + a\,h_3^2\,135\,z^4\,(1+4\,s+6\,s^2) + f_5^2\,h_3\,375\,s^2\,z^2\,(11+22\,z+3\,z^2) + a\,g_5\,h_3\,90\,z^3\,(5\,z^3 + 30\,z^2\,s + 8\,s^2\,(2+5\,z)) + b\,f_6\,h_3\,720\,z^2\,(2+2\,s\,(1+z^2) + s^2\,(7+3\,z^2)) + a^2\,b\,h_3\,432\,z^3\,(21\,z^2 + 21\,5\,s + 5\,s^2\,(1+6\,z))),$

$C[15,z,s] = z/5\,(b\,f_5\,n_3\,(3000\,z^3 + 4200\,z^3\,(1+z)s) + n_3^2\,45\,z^4\,s + b\,g_5\,h_3\;96\,z^3\,(7\,z^2 + 7\,5\,s + 40\,s^2) + a\,f_6\,n_3\,540\,z^3\,(z^2 + z\,4\,s + 12\,s^2) + b\,g_5^2\,80\,z^2\,(14\,z^3 + 70\,z^2\,s + 75\,s^2\,(1+2\,z)) + f_5\,f_6\,h_3\,1800\,z^2\,(z^2 + 4\,z^2\,s + 6\,s^2\,(1+2\,z)) + f_6^3\,4320\,s^2\,(1+2\,z+3\,z^2+4\,z^3) + f_5\,f_6\,g_5\,360\,z\,(13\,z^4 + 65\,z^3\,s + 50\,s^2\,(1+2\,z+3\,z^2)) + a_{24}\,h_4\,(80\,z^5 + 400\,z^5\,s) + a_{15}\,h_3\,(105\,z^4 + 15\,s\,z^4\,(23+5\,z)) + a_{05}\,h_4\,100\,z^4\,s^2 + a\,h_3\,h_4\,1800\,z^4\,s^2 + f_5^2\,h_4\,500\,z^2\,(6+6\,s\,(1+z) + 5\,z^2\,s^2) + a\,g_5\,h_4\,24\,z^3\,(28\,z^2 + 28\,5\,s\,z + 25\,s^2\,(12+5\,z)) + b\,f_6\,h_4\,960\,z^2\,(2\,z^4 + 12\,s + 3\,s^2\,z\,(4+z)) + a^3\,n_3\,3240\,z^4\,(z+4\,s+8\,s^2) + a^2\,f_5\,h_3\,10800\,z^3\,(z^3 + 3\,(1+s)\,s\,(1+z)) + a^2\,f_6^2\,25920\,s^2\,z\,(3+6\,z+9\,z^2+2\,z^3) + a^2\,f_5\,g_5\,720\,z^2\,(39\,z^3 + 5\,39\,s\,z^3 + 5\,s^2\,(30+60\,z+59\,z^2)) + a^2\,b\,h_4\,5760\,z^3\,(2+6\,s+s^2\,(4+13\,z)) + a^4\,f_6\,155520\,s^2\,z^2\,(3+6\,z+4\,z^2) + a^6\,933120\,s^2\,z^3\,(1+2\,z)),$

$C[16,z,s] = z^2\,(3\,a_{15}\,h_4\,z^4 + a_{24}\,b\,f_5\,1040\,z^3\,s + a\,a_{24}\,f_6\,936\,z^3\,s + a\,h_4^2\,24\,z^3\,(3+15\,s+8\,s^2) + f_5\,f_6\,h_4\,(29\,60\,z^2 + 60\,(3+s)\,s\,z\,(13+9\,z+8\,z^2)) + f_6^2\,g_5\,180\,(8\,(1+s\,z) + s^2\,(5+2\,z+7\,z^2+12\,z^3)) + b\,g_5\,h_4\,40\,z^2\,(9\,z^2+48\,s+4\,s^2\,(1+14\,z)) + a^2\,f_5\,h_4\,360\,z^2\,(29\,z^3 + s\,(52+52\,z+21\,z^2)) + f_5\,g_5^2\,125\,z\,(3+6\,s+2\,s^2\,(2+7\,z+12\,z^2)) + a^2\,f_6\,g_5\,2160\,z\,(8+3\,s+s^2\,(2+7\,z+12\,z^2)) + a^4\,g_5\,6480\,z^2\,(8\,z + s\,(1+5\,z) + 12\,s^2\,(1+z)) + a_{24}\,n_3\,(30\,z^4 + 36\,z^4\,(1+z)\,s) + f_5^2\,n_3\,75\,z^2\,(6+18\,s\,z^2 + s^2\,(7+14\,z+3\,z^2)) + a\,g_5\,n_3\,90\,z^3\,(1+s\,(3+z)\,(1+3\,s)) + f_5\,h_3^2\,45\,z^3\,(3+12\,s\,z+2\,s^2\,(5+4\,z)) + a\,h_3\,n_3\,54\,z^4\,(1+5\,s+7\,s^2) + b\,h_3\,h_4\,24\,z^3\,(9\,z+48\,s+4\,s^2\,(1+4\,z)) + b\,f_6\,n_3\,144\,z^2\,(6+18\,s\,z^2+s^2\,(7+14\,z+3\,z^2)) + f_6^2\,h_3\,108\,z\,(1+z)\,(4\,z^4 + 13\,s\,(1+z^2)) + f_5\,g_5\,h_3\,150\,z^2\,(3\,z^3 + 9\,s\,(1+z) + 4\,s^2\,(1+z)^2) + a^2\,b\,n_3\,864\,z^3\,(1+z)\,(3\,z^2 + 13\,s) + a^2\,f_6\,h_3\,1296\,z^2\,(8\,z^4 + s\,(13+13\,z+13\,z^2+4\,z^3)) + a^4\,h_3\,3888\,z^3\,(1+z)\,(4\,z^2 + 13\,s)),$

$C[17,z,s] = z^3\,(2100\,f_6\,g_5^2 + 12600\,a^2\,g_5^2\,z + 2520\,f_6\,g_5\,h_3\,z + 15120\,a^2\,g_5\,h_3\,z^2 + 756\,f_6\,h_3^2\,z^2 + 4536\,a^2\,h_3^2\,z^3 + a^2\,a_{24}\,b\,(3456\,z^3 + 4608\,z^3\,s\,(1+z)) + a_{24}\,b\,f_6\,(576\,z^2 + 768\,z^2\,s\,(1+z+z^2)) + a\,a_{24}\,g_5\,(360\,z^3 + 480\,z^3\,s\,(1+z)) + a_{24}\,f_5^2\,(300\,z^2 + 400\,z^2\,s\,(1+z+z^2)) + a\,a_{24}\,h_3\,(216\,z^4 + 288\,z^4\,s) + f_5\,h_3\,h_4\,(1680\,z^2\,s\,(1+z) + 960\,z^5) + b\,h_4^2\,128\,z^2\,(z^3 + 7\,s + 7\,s^2\,z) + f_6^2\,h_4\,(2016\,s\,(1+z+z^2+z^3) + 1152\,z^5) + f_5\,g_5\,h_4\,(2800\,z\,s\,(1+z+z^2) + 1600\,z^5) + a^2\,f_6\,h_4\,(24192\,z\,s\,(1+z+z^2) + 13824\,z^5) + a^4\,h_4\,(72576\,z^2\,s\,(1+z) + 41472\,z^5) + a^3\,a_{15}\,432\,s\,z^4\,(7+6\,z) + a_{15}\,b\,f_5\,(80\,z^3\,s\,(1+z) + 480\,z^3\,s\,(1+z+z^2)) + a\,a_{15}\,f_6\,(72\,z^3\,s\,(1+z) + 432\,z^3\,s\,(1+z+z^2)) + a_{15}\,n_3\,z^4\,s^2 + b\,h_3\,n_3\,24\,z^3\,(24\,z^3 + s\,(42+25\,z+17\,z^2)) + a\,h_4\,n_3\,24\,z^3\,(6\,z^4 + s\,(7+3\,z)\,(1+5\,s+2\,z^2)) + b\,g_5\,n_3\,40\,z^2\,(24\,z^4 + s\,(42+42\,z+25\,z^2+17\,z^3)) + f_5\,f_6\,n_3\,60\,z\,(24\,z^5 + s\,(42+42\,z+$

$$42\,z^2+25\,z^3+17\,z^4))+a^2\,f_5\,n_3\,360\,z^2\,(24\,z^4+s\,(42+42\,z+25\,z^2+17\,z^3))),$$

$$C[18,z,s]=\;z^4\,(625\,g_5^3+3600\,f_6\,g_5\,h_4+1125\,g_5^2\,h_3\,z+21600\,a^2\,g_5\,h_4\,z+2160\,f_6\,h_3\,h_4\,z+1200\,f_5\,h_4^2\,z+675\,g_5\,h_3^2\,z^2+12960\,a^2\,h_3\,h_4\,z^2+135\,h_3^3\,z^3+a^2\,a_{24}\,f_5\,(4680\,z^2+1080\,z^2\,s\,(1+z)+5040\,z^2\,s\,(1+z+z^2))+a^2\,a_{15}\,b\,(2880\,z^3\,s+1440\,z^3\,s\,(1+z))+a_{24}\,f_5\,f_6\,(780\,z+180\,z\,s\,(1+z+z^2)+840\,z\,s\,(1+z+z^2+z^3))+a_{24}\,b\,g_5\,(520\,z^2+120\,z^2\,s\,(1+z)+560\,z^2\,s\,(1+z+z^2))+a_{24}\,b\,h_3\,(312\,z^3+72\,z^3\,s+336\,z^3\,s\,(1+z))+a\,a_{24}\,h_4\,(312\,z^3+72\,z^3\,s+336\,z^3\,s\,(1+z))+a_{15}\,b\,f_6\,(480\,z^2\,s\,(1+z)+240\,z^2\,s\,(1+z+z^2))+a\,a_{15}\,g_5\,(300\,z^3\,s+150\,z^3\,s\,(1+z))+a_{15}\,f_5^2\,(250\,z^2\,s\,(1+z)+125\,z^2\,s\,(1+z+z^2))+a\,a_{15}\,h_3\,90\,z^4\,s+a_{15}\,a_{24}\,(27\,z^4+2\,z^5)+a\,n_3^2\,27\,z^3\,(1+2\,s)\,(1+4\,s)+f_5\,h_3\,n_3\,90\,z^2\,(1+z)\,(15\,s+4\,z^2)+b\,h_4\,n_3\,96\,z^2\,(1+z)\,(15\,s+4\,z^2)+f_6^2\,n_3\,108\,(1+z)\,(15\,s\,(1+z^2)+4\,z^4)+f_5\,g_5\,n_3\,150\,z\,(8\,z^4+s\,(15+15\,z+15\,z^2+4\,z^3))+a^2\,f_6\,n_3\,1296\,z\,(4+11\,s\,(1+z+z^2)+4\,z^4)+a^4\,n_3\,3888\,z^2\,(1+z)\,(15\,s+4\,z^2)+a_{05}\,a_{15}\,5\,z^4\,(2+z)\,(2+3s)),$$

$$C[19,z,s]=\;z^5\,(1600\,g_5^2\,h_4+1536\,f_6\,h_4^2+2880\,f_6\,g_5\,n_3+1920\,g_5\,h_3\,h_4\,z+9216\,a^2\,h_4^2\,z+17280\,a^2\,g_5\,n_3\,z+1728\,f_6\,h_3\,n_3\,z+1920\,f_5\,h_4\,n_3\,z+576\,h_3^2\,h_4\,z^2+10368\,a^2\,h_3\,n_3\,z^2+576\,b\,n_3^2\,z^2+a^4\,a_{24}\,20736\,(z^2+z^2\,s\,(1+z))+a^2\,a_{24}\,f_6\,6912\,(z+z\,s\,(1+z+z^2))+a_{24}\,f_5\,g_5\,800\,z\,(1+s\,(1+z+z^2))+a_{24}\,f_6^2\,576\,(1+s\,(1+z+z^2+z^3))+a_{24}\,b\,h_4\,512\,z^2\,(1+s\,(1+z))+a_{24}\,f_5\,h_3\,480\,z^2\,(1+s\,(1+z))+a\,a_{24}\,n_3\,288\,z^3\,(1+s)+a^2\,a_{15}\,f_5\,(2520\,z^2+360\,z^2\,s\,(1+z)+2880\,z^2\,s(1+z+z^2))+a_{15}\,f_5\,f_6\,(420\,z+60\,z\,s\,(1+z+z^2)+480\,z\,s\,(1+z+z^2+z^3))+a_{15}\,b\,g_5\,(280\,z^2+40\,z^2\,s\,(1+z)+320\,z^2\,s\,(1+z+z^2))+a_{15}\,b\,h_3\,(168\,z^3+24\,z^3\,s+192\,z^3\,s\,(1+z))+a\,a_{15}\,h_4\,(168\,z^3+24\,z^3\,s+192\,z^3\,s\,(1+z))),$$

$$C[20,z,s]=\;z^6\,(2040\,a_{24}\,f_6\,g_5+1360\,g_5\,h_4^2+1275\,g_5^2\,n_3+2448\,f_6\,h_4\,n_3+12240\,a^2\,a_{24}\,g_5\,z+1224\,a_{24}\,f_6\,h_3\,z+1360\,a_{24}\,f_5\,h_4\,z+816\,h_3\,h_4^2\,z+1530\,g_5\,h_3\,n_3\,z+14688\,a^2\,h_4\,n_3\,z+765\,f_5\,n_3^2\,z+7344\,a^2\,a_{24}\,h_3\,z^2+816\,a_{24}\,b\,n_3\,z^2+459\,h_3^2\,n_3\,z^2+204\,a\,a_{24}^2\,z^3+a^4\,a_{15}\,(10368\,z^2+6480\,z^2\,s\,(1+z)+5184\,z^2\,s\,(1+z+z^2))+a^2\,a_{15}\,f_6\,(3456\,z+2160\,z\,s\,(1+z+z^2)+1728\,z\,s\,(1+z+z^2+z^3))+a_{15}\,f_5\,g_5\,(400\,z+250\,z\,s\,(1+z+z^2)+200\,z\,s\,(1+z+z^2+z^3))+a_{15}\,f_6^2\,(288+180\,s\,(1+z+z^2+z^3)+144\,s\,(1+z+z^2+z^3+z^4))+a_{15}\,b\,h_4\,(256\,z^2+160\,z^2\,s\,(1+z)+128\,z^2\,s\,(1+z+z^2))+a_{15}\,f_5\,h_3\,(240\,z^2+150\,z^2\,s\,(1+z)+120\,z^2\,s\,(1+z+z^2))+a\,a_{15}\,n_3\,(144\,z^3+90\,z^3\,s+72\,z^3\,s\,(1+z))),$$

$$C[21,z,s]=\;1080\,a_{15}\,f_6\,g_5\,z^7+900\,a_{24}\,g_5^2\,z^7+1728\,a_{24}\,f_6\,h_4\,z^7+384\,h_4^3\,z^7+2160\,g_5\,h_4\,n_3\,z^7+972\,f_6\,n_3^2\,z^7+6480\,a^2\,a_{15}\,g_5\,z^8+648\,a_{15}\,f_6\,h_3\,z^8+1080\,a_{24}\,g_5\,h_3\,z^8+10368\,a^2\,a_{24}\,h_4\,z^8+720\,a_{15}\,f_5\,h_4\,z^8+1080\,a_{24}\,f_5\,n_3\,z^8+1296\,h_3\,h_4\,n_3\,z^8+5832\,a^2\,n_3^2\,z^8+288\,a_{24}^2\,b\,z^9+3888\,a^2\,a_{15}\,h_3\,z^9+324\,a_{24}\,h_3^2\,z^9+432\,a_{15}\,b\,n_3\,z^9+216\,a\,a_{15}\,a_{24}\,z^{10},$$

$$C[22,z,s]=\;475\,a_{15}\,g_5^2\,z^8+912\,a_{15}\,f_6\,h_4\,z^8+1520\,a_{24}\,g_5\,h_4\,z^8+1368\,a_{24}\,f_6\,n_3\,z^8+912\,h_4^2\,n_3\,z^8+855\,g_5\,n_3^2\,z^8+380\,a_{24}^2\,f_5\,z^9+570\,a_{15}\,g_5\,h_3\,z^9+5472\,a^2\,a_{15}\,h_4\,z^9+912\,a_{24}\,h_3\,h_4\,z^9+8208\,a^2\,a_{24}\,n_3\,z^9+570\,a_{15}\,f_5\,n_3\,z^9+513\,h_3\,n_3^2\,z^9+304\,a_{15}\,a_{24}\,b\,z^{10}+171\,a_{15}\,h_3^2\,z^{10}+57\,a\,a_{15}^2\,z^{11},$$

$$C[23,z,s]=\;480\,a_{24}^2\,f_6\,z^9+800\,a_{15}\,g_5\,h_4\,z^9+640\,a_{24}\,h_4^2\,z^9+720\,a_{15}\,f_6\,n_3\,z^9+1200\,a_{24}\,g_5\,n_3\,z^9+720\,h_4\,n_3^2\,z^9+2880\,a^2\,a_{24}^2\,z^{10}+400\,a_{15}\,a_{24}\,f_5\,z^{10}+480\,a_{15}\,h_3\,h_4\,z^{10}+4320\,a^2\,a_{15}\,n_3\,z^{10}+720\,a_{24}\,h_3\,n_3\,z^{10}+80\,a_{15}^2\,b\,z^{11},$$

$$C[24,z,s]=\;504\,a_{15}\,a_{24}\,f_6\,z^{10}+420\,a_{24}^2\,g_5\,z^{10}+336\,a_{15}\,h_4^2\,z^{10}+630\,a_{15}\,g_5\,n_3\,z^{10}+1008\,a_{24}\,h_4\,n_3\,z^{10}+189\,n_3^3\,z^{10}+3024\,a^2\,a_{15}\,a_{24}\,z^{11}+105\,a_{15}^2\,f_5\,z^{11}+252\,a_{24}^2\,h_3\,z^{11}+378\,a_{15}\,h_3\,n_3\,z^{11},$$

$$C[25,z,s]=\;132\,a_{15}^2\,f_6\,z^{11}+440\,a_{15}\,a_{24}\,g_5\,z^{11}+352\,a_{24}^2\,h_4\,z^{11}+528\,a_{15}\,h_4\,n_3\,z^{11}+396\,a_{24}\,n_3^2\,z^{11}+792\,a^2\,a_{15}^2\,z^{12}+264\,a_{15}\,a_{24}\,h_3\,z^{12},$$

$$C[26,z,s]=115\,a_{15}^2\,g_5\,z^{12}+368\,a_{15}\,a_{24}\,h_4\,z^{12}+276\,a_{24}^2\,n_3\,z^{12}+207\,a_{15}\,n_3^2\,z^{12}+69\,a_{15}^2\,h_3\,z^{13},$$

$$C[27,z,s]=\;64\,a_{24}^3\,z^{13}+96\,a_{15}^2\,h_4\,z^{13}+288\,a_{15}\,a_{24}\,n_3\,z^{13},$$

$$C[28,z,s]=\;100\,a_{15}\,a_{24}^2\,z^{14}+75\,a_{15}^2\,n_3\,z^{14},$$

$$C[29,z,s]=\;52\,a_{15}^2\,a_{24}\,z^{15},$$

$$C[30,z,s]=\;9\,a_{15}^3\,z^{16},$$

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